QUASISTATIC DEFORMATION OF A GRANULAR MEDIUM

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Static deformations of simple shear of a material composed of identical spherical particles are considered on the basis of maximal configurational entropy of the system macrostates. It is established theoretically and experimentally that the same principle holds also in slow shear flow of a material characterized by a universal value of the particle concentration which is independent of the initial concentration.

The number of natural phenomena and engineering applications related to the deformation and motion of loose materials is extremely great. The main problem arising in the theoretical description of such processes lies in the derivation of a physically substantiated system of fundamental rheological relations which would allow one to close the system of conservation equations for the indicated media. Owing to the objective complexity of the considered systems composed of a great number of particles and to quite significant difficulties associated with the description of their random behavior, the problem is still far from having been solved satisfactorily, even though most serious efforts have been made.

Attempts at a semiempirical solution of this problem usually involve traditional methods of plasticity theory [1] or simplified modeling of complex real displacements of the particles during deformation and flow with the aid of elementary shears in the slip planes and of rotation of these planes [2]. Such activity is oriented toward obtaining a phenomenological macroscopic law of the flow, in fact, without regard for the interaction between individual particles at the microlevel, which provokes extended discussions regarding the applicability of such an approach in principle [3, 4].

Since a continuum description does not permit one to trace either the displacements of individual particles or the resulting structural alterations in the granular medium, acceptance has been gained by interpretations of the behavior of a granular medium relying on the analysis of individual particles and their contacts (see, for example, [5, 6]). Without denying the correctness and usefulness of this approach, we should point out that, taken alone, it is far from being optimal for a noncontroversial formulation of the fundamental continuum equations. Here, the situation is very similar to that encountered in the mechanics of gases and other molecular and atomic systems of many particles: for all the intricacy of the behavior of separate molecules or atoms, the average states and motions of such systems are governed by relatively simple statistical laws and continuum equations.

Thus, it seems natural to apply certain general principles and hypotheses of statistical physics to the investigation of the problem. Such an approach was, supposedly, realized for the first time by A. M. Vaisman and M. A. Gol'dshtik [7]. The reasoning developed below largely exploits the methods of the above-mentioned study.

We consider randomly packed isotropic systems of identical rigid spheres interacting when in contact. In view of randomness and isotropy, the states of such systems may be defined by a single macroscopic parameter, viz., by the volume concentration of particles in the system φ. Without loss of generality, we may employ dimensionless quantities, assuming the diameter of each sphere to be equal to unity. Following [1], to model the system we consider two layers of particles, one atop the other; here we regard the lower layer as orderly at the nodes of a square lattice in contact with adjacent spheres (Fig. 1a), and the upper randomized. If the average distance between the plane of centers of the lower-layer spheres and the centers of the second-layer spheres (reckoned along the z axis in Fig. 1b) is equal to h, then

$$\varphi = \pi/(6h). \tag{1}$$

For the maximal and minimal values of h, equal to unity and $\sqrt{2}/2$, the corresponding concentrations are 0.524 and 740.

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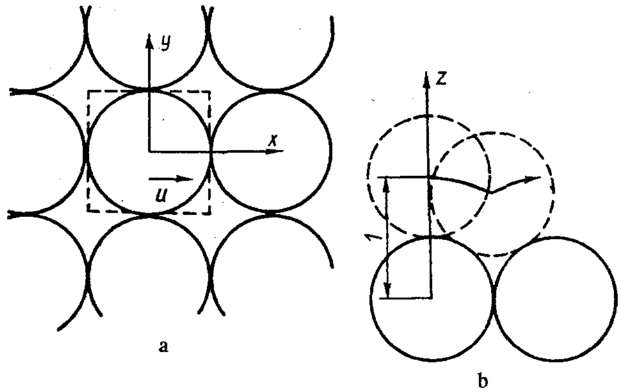


Fig. 1. Sketch of construction of a model: top view of an orderly lower row of spherical particles with indication of the direction of virtual displacement and of the region for determining the distribution function (a) and characteristic trajectory of the motion of an upper-row sphere with retained contact with lower-row spheres (b).

TABLE 1. Parameter β as a Function of h_0 and φ_0

β	30	20	10	5	1	0	—1	—5	—10	—20	—30
h_0	0,968	0,956	0,937	0,924	0,911	0,907	0,903	0,887	0,873	0,821	0,787
φ_0	0,540	0,547	0,558	0,566	0,574	0,577	0,579	0,590	0,599	0,637	0,665

We will assume that all physically acceptable positions of any upper-layer sphere, retaining at least one contact with the lower-layer spheres, are equally probable. It is then natural to search for the distribution function $\omega_0(r)$ of the sphere center in its horizontal deflections from the position with maximal height (Fig. 1b) based on the requirement of maximal configurational entropy of the macroscopic state, characterized by a specified average value of h [8]. Here, apparently, $r^2 = x^2 + y^2$, where the x and y axes are in the plane of centers of the lower-layer particles.

The configurational entropy of the system may be preset as [8]

$$S = \sum_i G_i n_i \ln (e/n_i),$$

where n_i is the number of particles and G_i is the statistical weight of the state characterized by the deflection r_i. Obviously, $G_i \sim h^{-1}(r_i)$, and the condition of conservation of the total number of particles N and of equality of the average height of the particle centers above the plane of the lower-layer centers to h_0 has the form

$$\sum_i G_i n_i = N, \quad \sum_i G_i n_i h(r_i) = h_0.$$

By introducing in the usual fashion the Lagrangian multipliers α and β, from the condition of entropy extremum as a function of n_i, we obtain

$$\omega_0(r_i) \sim n_i G_i = h^{-1}(r_i) \exp[\alpha + \beta h(r_i)].$$

By determining α and β from the normalization condition and the prescribed h_0 (or φ_0) and by expressing r in terms of x and y, we eventually arrive at

$$\omega_0(r) = \frac{\exp(\beta\sqrt{1-r^2})}{A\sqrt{1-r^2}}, \quad A = \iint \frac{\exp(\beta\sqrt{1-r^2})}{\sqrt{1-r^2}} dxdy, \tag{2}$$

here for β there is the equation

$$\frac{1}{A}\iint \exp(\beta\sqrt{1-r^2})\, dxdy = h_0 = \frac{\pi}{6\varphi_0}. \tag{3}$$

The integration region in the above equations is determined by reasons of symmetry and is of square shape in Fig. 1a. Outside the square, it may be constructed proceeding from its periodicity along the x and y axes with the period $2\Delta = 1$.

Table 1 illustrates the parameters β and h_0 as functions of φ_0.

We note that the study [7] incorrectly identified the values of r_i with the deflections x_i along one of the axes. This corresponds to the situation with deformation of a system of identical cylindrical particles in the plane normal to their generatrices (recently, such systems have been studied quite intensely) but not of a real granular medium with spherical particles.

It should also be noticed that Eq. (2) is derived neglecting the positions of neighboring spheres of the upper layer, which, generally speaking, is impermissible when describing high-concentration systems. For describing such systems, along with ordinary virial expansions there are a number of approximate models of the statistical physics of dense gases and liquids, which are also poorly suited to the analysis of concentrated systems. However, such an approach might be justified by a certain arbitrariness of the model considered and by its qualitative character; efforts to introduce correlations of particles into the analysis would hardly be appropriate at this stage of the research.

Now we examine the variation in $\omega(r)$ on displacement of all spheres of the upper row along the x axis, with contact between the upper and lower spheres retained. In principle, it is possible to study the consequences of linear displacement over the distance u, which is the same for all spheres, when the effective values of the shear $\gamma(r) = u/h(r)$ are different, or the centers of different spheres are displaced over different distances $u(r) = \gamma h(r)$ at $y = \text{const}$ (as in [17]). In the first case, for $u = 2\Delta$ we, evidently, arrive again at the initial state, i.e., any characteristics of the disturbed state as functions of u change periodically with the period 2Δ. In the second case, return to the initial state appears, as is easy to see, to be impossible. Since the experimental data (see the subsequent discussion) testify to the periodic variation of the state, we perform basic computations with reference to the first variant.

We now point out that the study [7] examined displacement of diagonal chains parallel to themselves instead of displacement of chains in the direction of the lattice face. Because of the qualitative character of the model this distinction is quite insignificant.

Clearly, in the statement considered, the distribution function $\omega(x, y; u)$ of the disturbed state in horizontal deflections from a position of maximal height is governed entirely by the initial distribution $\omega_0(r)$. There is no difficulty in representing this function, with allowance for Eq. (2), as

$$\omega(x, y; u) = \frac{\exp(\beta\sqrt{1-(x+u)^2-y^2})}{A\sqrt{1-(x+u)^2-y^2}}, \quad 0 < u < \Delta - x,$$
$$\omega(x, y; u) = \frac{\exp(\beta\sqrt{1-(2\Delta-x-u)^2-y^2})}{A\sqrt{1-(2\Delta-x-u)^2-y^2}}, \quad \Delta - x < u < 2\Delta \tag{4}$$

(for definiteness u is assumed positive), where the normalization parameter A is, as before, determined by Eq. (2), and β is the only root of Eq. (3). In Eq. (4), $r^2 = x^2 + y^2$.

The average values of h(u) and of the concentration $\varphi(u)$ for the disturbed state may be defined as

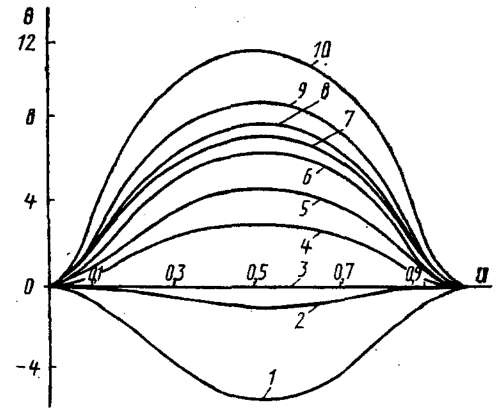
$$h(u) = \iint\sqrt{1-x^2-y^2}\,\omega(x, y; u)\,dx dy,$$
$$\varphi(u) = \pi/(6h(u)), \tag{5}$$

they represent functions of u as well as of h_0 (or φ_0). In a similar way we may also determine the average values of any other state function, for example, the average shear:

$$\gamma(u) = u\iint\omega(x, y; u)\frac{dx dy}{\sqrt{1-x^2-y^2}}. \tag{6}$$

The volume strain may be expressed in the form

$$\theta = (h_0 - h(u))\,h^{-1}(u). \tag{7}$$



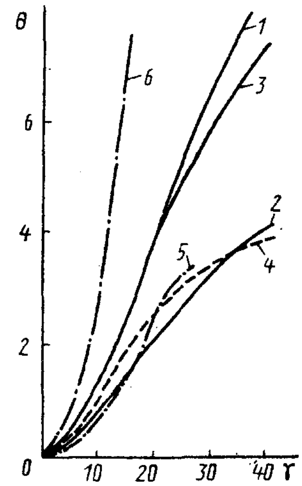


Fig. 2

Fig. 3

Fig. 2. Volume strain vs displacement with β determined from Eq. (3): curves 1-10 correspond to initial concentrations of 0.537, 0.560, 0.577, 0.590, 0.604, 0.617, 0.623, 0.630, 0.635, and 0.660. θ, %.

Fig. 3. Volume strain of a granular medium in static shear conditions: 1, 2) model of displacement with variable β (explanations are given in the text); 3) the same with constant β; 4) experimental data from [10]; 5, 6) curve presented in [7] and its shape when correctly calculated; 1, 3, 5, 6) φ = 0.630; 2) 0.604; 4) 0.640. γ, %.

Figure 2 gives characteristic calculated results for θ as a function of u for various initial concentrations. If the initial concentration is $\varphi_0 > 0.577$ ($h_0 < 0.907$), shear strain leads to an increase in the specific volume of the particles of the medium, i.e., to expansion of the medium. Otherwise, the granular medium is densified when deformed. The medium volume with $\varphi_0 = 0.577$ is preserved without change during deformation.

Figure 3 presents calculated results for θ from Eq. (7) as a function of γ from Eq. (6), together with the known experimental curve of Taylor [10] for a dense sand ($\varphi_0 = 0.640$). The same figure shows a theoretical curve of [7] associated with displacement of the lower layer particles along a diagonal of the lattice for γ = const, rather than in the face direction, as in Fig. 1a. Here, as indicated above, the dependence of the distribution function on the coordinate normal to the diagonal direction was disregarded. However, we have not managed to repeat the calculation of this curve; instead, with the above-mentioned coordinate ignored (i.e., actually for the system of parallel cylindrical fibers displaced over the distance $u(x_i) = \gamma h(x_i)$), for γ = const we have obtained a very steep curve $\theta(\gamma)$, which is also given in Fig. 3 and obviously does not check with the experimental data. For fairly small shears, the theoretical results are in agreement with the experimental data, which is not the case with increasing shear; here the theoretical curves lie much higher than the experimental one. The values of shear strain along the abscissa are given in percent of u to the initial height h_0.

This discrepancy is likely connected with the premise that the lower layer of particles is orderly and with the corresponding neglect of its randomization, indispensable in the conditions of finite deformations, and, probably, with the fact that sand particles of irregular shape were used in Taylor's experiments, rather than identical spheres. Within the framework of the calculational scheme outlined, such randomization is disregarded, and the granular medium retains the memory of the initial state for an indefinitely long period of time. In reality, of course, the lower layer of spheres acts as an upper one for the underlying layer, and its statistical characteristics should not differ radically from those of the upper layer. Thus, irrespective of the initial state of the granular medium, its deformation must involve additional random plastic strains linked with redistribution of the particles which are just the ones to ensure the randomization mentioned above.

The plastic strains result in the fact that a certain averaged behavior of the granular medium, not depending any longer on the initial state, should be observed instead of the periodic variation in properties according to the curves in Fig. 2. Since random media inevitably cause a loss of memory of properties of the initial state, properties of the state

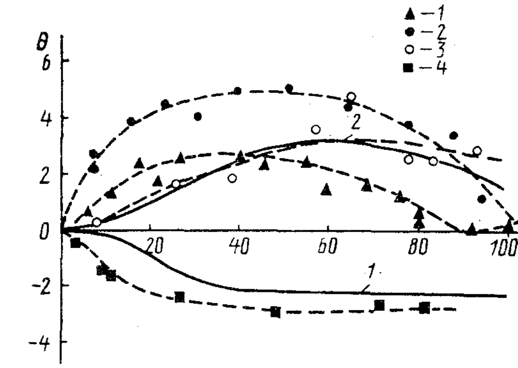


Fig. 4. Volume strain vs shear magnitude; experimental points: 1, 2) two layers of particles at φ_0 = 0.550 and 0.583, respectively, 3, 4) seven layers of particles, φ_0 = 0.560 and 0.522; calculated curves: 1) at variable β and φ = 0.560, 2) at constant β and φ_0 = 0.590.

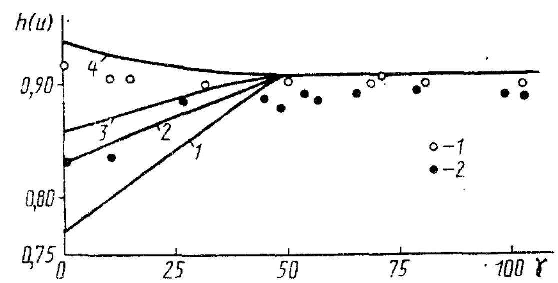


Fig. 5. Establishment of a limiting steady state according to the model at φ_0 = 0.670, 0.629, 0.580, and 0.560 (curves 1-4, respectively); experimental points 1 and 2 correspond to deep filling at φ_0 = 0.560 and 0.522, respectively.

achieved during a prolonged and fairly slow shear flow must be universal and time-independent. The exception may be only the situations where the plastic strain is hindered for one reason or another, as is the case, for example, with a very small number of particle layers between rigid walls confining the medium.

If the shear flow proceeds with a rather low velocity, it is natural to employ the idea of quasi-equilibrium of the medium at any instant of time as the working hypothesis for the phenomenological description of properties of the indicated limiting state of the deformed medium. This implies that any intermediate state should also satisfy the principle of maximal configurational entropy at the characteristic value of h(u). In such a situation, for the particle distribution function in deflections from the state with a maximal height, Eq. (1) is preserved with the previous definition of the normalization constant, but the quantity β must now be determined from Eq. (3), where the running h_u is used in lieu of the initial value h_0. At the same time, the distribution function in the state in question may also be considered as that obtained from Eq. (4) as a result of evolution of the state, with the given h(u) regarded as initial. This means that, as applied to the ultimate limiting state, the results calculated from Eqs. (1) and (2) must merely coincide. The requirement of such coincidence might be used as the condition determining the indicated limiting state. The corresponding curves computed from Eq. (1) are also given in Fig. 3. As is seen easily, the periodic variations in properties of the granular medium are actually lost.

Some results of computations according to models with constant and variable β are presented in Figs. 4 and 5. The analysis of these figures confirms, on the whole, the idea of the granular medium forgetting its initial state. Besides, it follows from the character of the curves in Fig. 5 that the same stabilized state is in fact attained for any initial state, i.e., it is really universal. The particle concentration in this limiting state is exactly the same as the concentration of the granular medium not subjected to volume compression or expansion.

To check the reasoning set forth we conducted special experiments. We used an annular measuring cell of the same type as in the experiments of [11, 12]. The cell was confined in the vertical direction by two concentric disks made of organic glass, which permitted visual observations. The lower disk was attached rigidly to a rod with the aid of a connecting sleeve, through which it was actuated by an electric motor with a rotation frequency of 0.25 rpm. An annular groove, where the material considered was placed, was made in the lower disk. The thickness of the groove was 0.04 m and the average diameter 0.176 m. The upper disk was fastened to the rod via an attached bush, being a sliding bearing, and had an annular bulge that could move in the groove, of size smaller than the groove width by not more than 0.001 m. The end surface of the upper disk was lowered down the material filling the groove and pressed by a weight. During the experiments the upper disk was kept from rotating by an immovable beam free to move vertically. To prevent slip of the particles on the shear planes, one layer of grains of the same size as the tested particles was affixed to the groove bottom and to the bulge surface. The displacement of the upper disk from the lower one was measured by displacement indicators and tensometric sensors glued to flexible plates with one end attached rigidly to the fixed racks and the other descending freely to the surface of the upper disk. The tensometric sensors were connected through an amplifier to self-recorders.

As the granular medium we used monofractions of glass spheres and of iron-ore pellets of average radii 0.0032 and 0.01255 m, respectively. The concentration of the charge of height of several particle diameters was determined based on familiar methods [11] from the mass and density of separate particles and from the groove volume occupied by the loose material. In situations with only two rows of particles located in the groove the concentration was calculated from relation (1). It should be remarked that, because the real packing of the particles of a layer differs from a regular square one, the relationship between the concentration and the average spacing of particles in adjacent displaced layers corresponds to the formula

$$\varphi_0 = \pi/(kh_0),$$

where the parameter k is not equal, generally speaking, to six (as in Eq. (1)) but is readily determined from experiments. Use of the above formula (1) was dictated by the circumstance that, owing to the wall effect, the particles cannot be packed closely, as in tanks of infinite size, and conventional methods are inapplicable.

In the case of two horizontal rows of particles, the memory of the initial state is obviously retained, and the evolution of the system properties with increasing shear strain is periodic or at least nearly such in character, although there is no complete quantitative agreement with theoretical results (Fig. 4). The latter fact is quite understandable in the context of the qualitative nature of the model with constant β, underlying the computations. More importantly, regardless of the initial concentration the granular medium indeed arrives at a state of the same concentration: all experimental points lie in the narrow range $0.89 < h_\infty < 0.92$. These deductions fully verify the hypothesis of a quasiequilibrium state of the granular medium achieved during its quasistatic deformation. Moreover, this hypothesis, even when applied within the framework of a fairly simple approximate model, leads to quite satisfactory quantitative results. In particular, the model allows one to describe with sufficiently high accuracy the classical Reynolds effect, i.e., densification of the deformed medium at small initial concentrations, corresponding to h_∞ = 0.89-0.92, and its loosening in the opposite case.

If the deformation proceeds at a significant or, in any case, finite rate, the steady states of the medium achieved here must differ from quasiequilibrium more strongly, the higher this rate. However, in this case also we may hope that the last state will prove to be a convenient "reference point" in constructing an expanded rheological law of the granular-medium flow depending on the value of the shear velocity.

NOTATION

φ, volume concentration of the particles; h, average distance between the plane of centers of the lower-layer spheres and the centers of the upper-layer spheres; $\omega(r)$, distribution function; r, horizontal deflection; S, configurational entropy; n, number of particles; G, statistical weight; N, total number of particles in the system; α, β, Lagrangian multipliers; 2Δ, spacing of the particle lattice; u, horizontal displacement; γ, average shear.

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